The existence of a near-unanimity term in a finite algebra is decidable

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The NU problem for finite algebras

Definition

An n-ary operation f on a set A is a *near-unanimity operation*, if $n \geq 3$ and

$$f(y, x, \dots, x) = f(x, y, x, \dots, x) = \cdots = f(x, \dots, x, y) = x$$

for all $x, y \in A$.

Problem

Instance: a finite algebra $\langle A; \mathcal{F} \rangle$ of finite signature
Question: does $\langle A; \mathcal{F} \rangle$ have a near-unanimity term operation?

The set of all n-ary operations in the clone $\langle \mathcal{F} \rangle$ can be easily computed, but we do not know how large n must be to find the near-unanimity operation.

Background and results

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Theorem (M. Maróti, 2005)

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- Show that using iterated composition we get an increasing sequence of order filters, each of which can be represented with their (finitely many) minimal elements.
- Since the set of order filters satisfies the ascending chain condition, the procedure must stop, and the closure can be computed.

A is a fixed finite set, $\mathcal{O}^{(n)} = A^{A^n}$, $\mathcal{O} = \bigcup_{n\in\omega} \mathcal{O}^{(n)}$, and $\omega = \{0, 1, 2, \dots\}$.

Definition

Let $f \in \mathcal{O}^{(n)}$ and $i \in \omega$. The *i*-th polymer of f is $f|_i \in \mathcal{O}^{(2)}$ defined as

$$f|_i(x,y) = \begin{cases} f(x,\dots,x,\overset{i}{\smile}{y},x,\dots,x) & \text{if } i < n, \\ f(x,\dots,x) & \text{if } i \geq n. \end{cases}$$

The characteristic function of f is the map $\chi_f : \mathcal{O}^{(2)} \to \omega + 1$ defined as

$$\chi_f(b) = |\{\, i \in \omega : f|_i = b \,\}|\,.$$

Example

If $\pi \in \mathcal{O}$ is a projection and $\nu \in \mathcal{O}$ is a near-unanimity operation, then

$$\chi_\pi(b) = \begin{cases} 1 & \text{if } b(x,y) = y, \\ \omega & \text{if } b(x,y) = x, \\ 0 & \text{otherwise,} \end{cases} \qquad \chi_\nu(b) = \begin{cases} \omega & \text{if } b(x,y) = x, \\ 0 & \text{otherwise.} \end{cases}$$

Composition

Motivation

Let $f \in \mathcal{O}^{(n)}$ and $g_0, \ldots, g_{n-1} \in \mathcal{O}^{(m)}$. What information do we need to compute the characteristic function of $f(g_0, \ldots, g_{n-1})$?

- the characteristic functions of $g_0, \ldots, g_{n-1}$
- the operation f
- "assignment of variables"

We denote the set of all characteristic functions by $\mathcal{X}$.

Definition

We say that $\chi \in \mathcal{X}$ is a composition of $f \in \mathcal{O}^{(n)}$ and $\chi_0, \ldots, \chi_{n-1} \in \mathcal{X}$ if there exists a map $\mu : (\mathcal{O}^{(2)})^n \to \omega + 1$ such that for all $b \in \mathcal{O}^{(2)}$ and $i < n$

$$\chi(b) = \sum_{\bar{c} \in (O^{(2)})^n,\ f(\bar{c}) = b} \mu(\bar{c}), \qquad \text{and} \qquad \chi_i(b) = \sum_{\bar{c} \in (O^{(2)})^n,\ c_i = b} \mu(\bar{c}).$$

Definition

We say that $h \in \mathcal{O}^{(m)}$ is a composition of $f \in \mathcal{O}^{(n)}$ and $g_0, \dots, g_{n-1} \in \mathcal{O}$ of arities at most m, if each g_i can be extended to an m-ary operation g_i' by introducing dummy variables and permuting the variables such that $h = f(g_0', \dots, g_{n-1}')$.

Definition

Let $\mathcal{F}, \mathcal{G} \subseteq \mathcal{O}$ and $\mathcal{Y} \subseteq \mathcal{X}$. We define the following operators

$$\begin{aligned} \mathbb{X}(\mathcal{G}) &= \{\, \chi_g : g \in \mathcal{G} \,\}, \\ \mathbb{C}_{\mathcal{F}}(\mathcal{G}) &= \{\, \text{compositions of } f \in \mathcal{F} \cap \mathcal{O}^{(n)} \text{ and } g_0, \dots, g_{n-1} \in \mathcal{G} \,\}, \\ \mathbb{C}_{\mathcal{F}}(\mathcal{Y}) &= \{\, \text{compositions of } f \in \mathcal{F} \cap \mathcal{O}^{(n)} \text{ and } \chi_0, \dots, \chi_{n-1} \in \mathcal{Y} \,\}. \end{aligned}$$

Lemmas

- $\mathbb{X}\mathbb{C}_{\mathcal{F}}(\mathcal{G}) = \mathbb{C}_{\mathcal{F}}\mathbb{X}(\mathcal{G})$ for all $\mathcal{F}, \mathcal{G} \subseteq \mathcal{O}$.
- If $\mathcal{F} \subset \mathcal{O}$ and $\mathcal{Y} \subset \mathcal{X}$ are finite, then so is $\mathbb{C}_{\mathcal{F}}(\mathcal{Y})$.

An interesting operation

Lemma

Let $\mathcal{C}$ be a clone on an m-element set. If $\mathcal{C}$ contains a near-unanimity operation, then it contains an operation g of arity at most $2^{m!} + m^{m^2}$ such that

$$\chi_g(b) = \begin{cases} \omega & \text{if } b(x,y) = x, \\ 2^{m!} & \text{if } b = c, \\ 0 & \text{otherwise} \end{cases}$$

where c is a binary operation that satisfies $c(x, c(x, y)) = c(x, y)$.

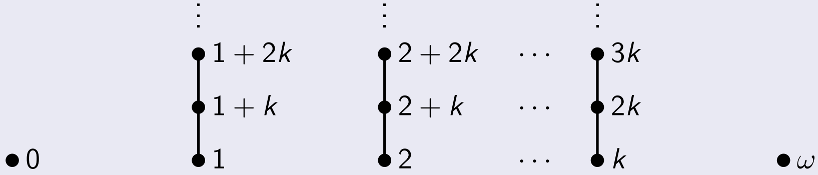
Theorem (L. Lovász, 1978)

Let n, k be natural numbers such that $2 \leq 2k \leq n$, and $G_{n,k}$ be the graph on the set of all k-element subsets of an n-element set where two subsets are connected if they are disjoint. The chromatic number of $G_{n,k}$ is $n - 2k + 2$.

The partial order

Definition

Put $k = 2^{|A|!} - 1$, and consider the following partial order $\sqsubseteq$ on $\omega + 1$:



Fact

The partial order $\sqsubseteq$ on $\mathcal{X}$ applied coordinate-wise has no infinite anti-chain and satisfies the descending chain condition. Therefore, the set of order filters under inclusion satisfies the ascending chain condition.

Lemma

$\uparrow \mathbb{C}_{\mathcal{F}}(\mathcal{Y}) = \mathbb{C}_{\mathcal{F}} \uparrow(\mathcal{Y})$ *for all* $\mathcal{F} \subseteq \mathcal{O}$ *and* $\mathcal{Y} \subseteq \mathcal{X}$.

Decision procedure

- Search for an interesting operation $g \in \langle \mathcal{F} \rangle$. If there is none, then $\langle \mathcal{F} \rangle$ has no NU operation.
- Then find the least natural number m such that

$$\mathbb{C}_{\mathcal{F}}^{m+1}(\{\chi_g\}) \subseteq \uparrow \bigcup_{n=0}^{m} \mathbb{C}_{\mathcal{F}}^{n}(\{\chi_g\})$$

- Finally, $\langle \mathcal{F} \rangle$ has a NU operation if and only if

$$\chi_\nu \in \bigcup_{n=0}^{m} \mathbb{C}_{\mathcal{F}}^{n}(\{\chi_g\}).$$

Key step

There exists a set $\{g\} \subseteq \mathcal{G} \subseteq \langle g \rangle$ such that $\mathbb{X}(\mathcal{G}) = \uparrow(\{\chi_g\})$, and

$$\mathbb{X} \bigcup_{n \in \omega} \mathbb{C}_{\mathcal{F}}^{n}(\mathcal{G}) = \bigcup_{n \in \omega} \mathbb{C}_{\mathcal{F}}^{n} \mathbb{X}(\mathcal{G}) = \bigcup_{n \in \omega} \mathbb{C}_{\mathcal{F}}^{n} \uparrow(\{\chi_g\}) = \bigcup_{n \in \omega} \uparrow \mathbb{C}_{\mathcal{F}}^{n}(\{\chi_g\}).$$

Conclusion

Corollary

It is decidable for a finite algebra in a congruence distributive variety whether it admits a natural duality.

Open Problem (B. A. Davey and R. McKenzie)

Given a finite algebra, decide if it admits a natural duality.

Open Problem (M. Y. Vardi)

Given a finite set of relations over a finite set, decide if there exists a compatible near-unanimity operation.

Open Problem (M. Valeriote)

Is it true that every clone on a finite set that contains a near-unanimity operation also contains a ternary weak near-unanimity operation, that is, an operation f satisfying $f(y,x,x) = f(x,y,x) = f(x,x,y)$.